ISRAEL JOURNAL OF MATHEMATICS, Vol. 71, No. 1, 1990

A NON-GEOMETRIC SHELLING OF A 3-POLYTOPE

BY

ZEEV SMILANSKY

Orbot Systems Ltd., 70651 Yavne, Israel

ABSTRACT

We describe a shelling of the octahedron in E^3 which cannot be realized in the conventional geometric way.

1. Introduction

Let P be a convex d-polytope with n facets in E^d. A *shelling* of the boundary complex $\mathcal{C}(P)$ of P is an arrangement $F_1, F_2, \ldots, F_n$ of the facets of P such that $F_i \cap (\bigcup_{j<i} F_j)$ is a topological $(d-2)$-ball for all i, $1 < i < n$. (In any arrangement of the facets, $F_n \cap (\bigcup_{j<n} F_j) = \mathrm{bd} F_n$ is a $(d-2)$-sphere.) The question of whether every polytope admits a shelling was considered very difficult for a time (see the remarks of Grünbaum in [3, pages 141–142]). However, when the answer came, it was surprisingly simple. In fact, as proved by Bruggesser and Mani in 1971 [2], the boundary complex of any convex polytope P is shellable in a particularly simple, geometric way: assume that $0 \in \mathrm{int} P$, write $H_i = \mathrm{aff} F_i = \{x \in E^d : \langle u_i, x \rangle = 1\}$ and pick $u \in S^{d-1}$ that satisfies $\langle u_i, u \rangle \neq \langle u_j, u \rangle$ whenever $i \neq j$ and $\langle u_i, u \rangle \neq 0$ for all $i = 1, \ldots, n$. Arranging the numbers $\alpha_i = \langle u_i, u \rangle$ by increasing order $\alpha_{i_1} < \alpha_{i_2} < \cdots \leq 0 < \cdots < \alpha_{i_n}$, the corresponding sequence of facets $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ is a shelling. Let us call such a shelling a *geometric shelling*. We shall also call a shelling *geometric* in the following, more general situation: $F_1, F_2, \ldots, F_n$ is a shelling of the facets of a polytope P, and a polytope Q exists which is combinatorially equivalent to P, and for which the corresponding sequence of facets of Q is a geometric shelling. If a shelling is not geometric, we say that it is a *non-geometric shelling*. Thus, a shelling of P is non-geometric

Received November 27, 1989

if and only if for every polytope Q, combinatorially equivalent to P, the shelling of Q induced by the original shelling of P is not a geometric shelling.

It is known (though unpublished) that non-geometric shellings exist in dimensions at least four, and do not exist in dimensions smaller than three. In this paper we describe a non-geometric shelling of the three-dimensional octahedron, thereby proving that non-geometric shellings exist in all dimensions at least three, and only there.

We shall use in this paper a process dual to shelling. The concept is useful for geometric shellings of general polytopes as well as for general shellings of simplicial polytopes.

DEFINITION 1.1. Let P be a d-polytope in E^d such that $0 \in \operatorname{int} P$. An ordering $v_1, v_2, \ldots, v_n$ of the vertices of P will be called a *peeling of* vert P if and only if the corresponding sequence $F_1, F_2, \ldots, F_n$ of the facets of the polar P^* of P is a shelling of bd$P$. A peeling will be called *geometric* if and only if the corresponding shelling is geometric. Thus, a peeling is geometric if, and only if, for some geometric realization of P, there is a $u \in S^{d-1}$, such that $\langle u, v_i \rangle < \langle u, v_j \rangle \Leftrightarrow i < j$.

REMARK. An observation of G. Kalai ([5]) implies that an ordering of the vertices of a simple d-polytope is a peeling order if, and only if, the directed graph on the skeleton, induced by the peeling order on the vertices, has the property that every face has a unique local maximum.

2. Geometric and non-geometric shellings

We shall now prove that non-geometric shellings exist in dimensions at least three. Concerning lower dimensional polytopes, the problem is meaningless for $d < 2$, while for 2-polytopes it is easy to construct, for any shelling of a k-gon, a (convex) k-gon that admits this shelling geometrically. Let us turn our attention to the case $d > 2$. To make the discussion complete, we start by describing a known example of a non-geometric shelling of a 4-polytope. First we need a definition.

DEFINITION 2.1. Let P be a d-polytope in E^d and q a point in E^d. For a point $p \in P$ we say that q *sees* p if, and only if, $[q,p] \cap P = \{p\}$. If $q \in E^d \setminus P$ then the set $\{p \in P : q \text{ sees } p\}$ is a union of (closed) facets of P that is homeomorphic to a $(d-1)$-ball. Consider a set $\mathfrak{F} = \{F_{i_1}, F_{i_2}, \ldots, F_{i_k}\}$ of facets of P. Let us call $\mathfrak{F}$ a *bad set of facets* if $\cup\mathfrak{F}$ is homeomorphic to a $(d-1)$-ball, but for no realization of P can a point $q \in E^d$ be found that sees precisely those facets of P that are in $\mathfrak{F}$.

EXAMPLE 1. In [4], Grünbaum and Sreedharan construct a simplicial 4-polytope P (P_5^7 in the original paper) with 7 vertices, 14 facets and a bad set of 4 facets. To be precise, label the vertices by indices 1–7. Citing [4], the names and vertex sets of the facets of P are as follows:

$$A:1234 \quad B:1237 \quad C:1267 \quad D:1256 \quad E:1245 \quad F:1347 \quad G:1457$$

$$H:1567 \quad J:2345 \quad K:2356 \quad L:2367 \quad M:3467 \quad N:3456 \quad O:4567$$

The bad set of facets is $\{D,E,G,K\}$, and it is easy to construct a shelling that starts with these four facets (example: $G,E,D,K,H,J,A,C,F,B,L,M,N,O$). It follows that this shelling is non-geometric. (The reader is referred to [4] for more information, including a Schlegel diagram of P.) ■

The polytope in Example 1 admits a non-geometric shelling because it has a bad set of facets. In 3 dimensions, a set of facets is bad if and only if its boundary, which is a simple circuit of edges, is not realizable as a shadow boundary of some projection. However, as proved by Barnette ([1]), given a simple circuit of edges on the boundary of a 3-polytope P, there exists a 3-polytope Q, combinatorially equivalent to P, such that the corresponding circuit in Q is the shadow boundary of some (parallel) projection. Hence it is natural to wonder whether there are non-geometric shellings in 3 dimensions. The following example shows that indeed there are, even in the simple case of peeling the vertices of the 3-cube.

EXAMPLE 2. Let Q be a 3-cube, with its vertices labeled by their peeling order, as indicated in Fig. 1.

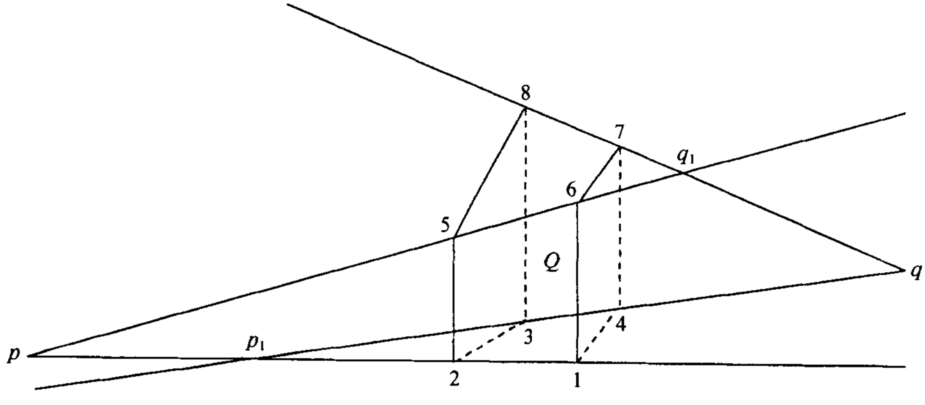

Figure 1

It is easy to verify that the ordering 12345678 is indeed a peeling of vert Q (consider Q^*, or use the remark following Definition 1.1). Suppose now that a realization of Q as a convex polytope and a series $H_1, H_2, \ldots, H_8$ of parallel planes exist in E^3, such that H_i contains the vertex i and separates $1,2,\ldots,i-1$ from $i+1, i+2, \ldots, 8$. Consider the facet $F_1 = \mathrm{conv}(1256)$, and the lines $l_2 = H_2 \cap F_1$ and $l_5 = H_5 \cap F_1$. Since l_2 and l_5 are assumed to be parallel, the lines $\overline{12}$ and $\overline{56}$ ($\overline{ij}$ denotes the line through vertices i and j) must meet at some point p beyond aff(2358). Similarly, $p_1 = \overline{12} \cap \overline{34}$ lies beyond aff(2358), and both $q = \overline{34} \cap \overline{78}$ and $q_1 = \overline{56} \cap \overline{78}$ lie beyond aff(1467). We have:

$$\overline{pq} = \mathrm{aff}(1234) \cap \mathrm{aff}(5678), \ \overline{p_1 q_1} = \mathrm{aff}(1256) \cap \mathrm{aff}(3478).$$

Since Q is assumed to be a convex polytope,

$$\begin{aligned} \overline{pq} \bigcap Q &= (\mathrm{aff}(1234) \cap Q) \cap (\mathrm{aff}(5678) \cap Q) \\ &= \mathrm{conv}(1234) \bigcap \mathrm{conv}(5678) = \varnothing, \end{aligned}$$

and in particular,

$$\overline{pq} \cap \mathrm{conv}(1234) = \varnothing.$$

This can happen only when $\overline{pq}$ intersects $\overline{12}$ outside the closed interval $[1, p_1]$, or, in other words, only when $p_1 \in \mathrm{int}[p, 2]$. But a similar argument, applied to conv(1256), shows that

$$\overline{p_1 q_1} \cap \mathrm{conv}(1256) = \varnothing,$$

and so $p \in \mathrm{int}[p_1, 2]$. This is a contradiction, and it follows that such a configuration cannot exist. ■

Acknowledgement

The author is indebted to David Barnette for helpful discussions and references during the preparation of this manuscript.

References

1. D. W. Barnette, *Projections of 3-polytopes*, Isr. J. Math. **8**(1970), 304–308.

2. H. Bruggesser and P. Mani, *Shellage decompositions of cells and spheres*, Math. Scand. **29** (1971), 197–205.

3. B. Grünbaum, *Convex Polytopes*, Wiley-Interscience, New York, 1967.

4. B. Grünbaum and V. P. Sreedharan, *An enumeration of simplicial 4-polytopes with 8 vertices*, J. Comb. Theory **2** (1967), 437–465.

5. G. Kalai, *A simple way to tell a simple polytope from its graph*, J. Comb. Theory, Ser. A **49** (1988), 381–383.